# Answers to Odd-Numbered Exercises

## CHAPTER 1

### Section 1.1

**1. a)** Yes, T **b)** Yes, F **c)** Yes, T **d)** Yes, F **e)** No **f)** No **3. a)** Today is not Thursday. **b)** There is pollution in New Jersey. **c)** $2 + 1 \neq 3$. **d)** The summer in Maine is not hot or it is not sunny. **5. a)** Sharks have not been spotted near the shore. **b)** Swimming at the New Jersey shore is allowed, and sharks have been spotted near the shore. **c)** Swimming at the New Jersey shore is not allowed, or sharks have been spotted near the shore. **d)** If swimming at the New Jersey shore is allowed, then sharks have not been spotted near the shore. **e)** If sharks have not been spotted near the shore, then swimming at the New Jersey shore is allowed. **f)** If swimming at the New Jersey shore is not allowed, then sharks have not been spotted near the shore. **g)** Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore. **h)** Swimming at the New Jersey shore is not allowed, and either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore. (Note that we were able to incorporate the parentheses by using the word "either" in the second half of the sentence.) **7. a)** $p \wedge q$ **b)** $p \wedge \neg q$ **c)** $\neg p \wedge \neg q$ **d)** $p \vee q$ **e)** $p \rightarrow q$ **f)** $(p \vee q) \wedge (p \rightarrow \neg q)$ **g)** $q \leftrightarrow p$ **9. a)** $\neg p$ **b)** $p \wedge \neg q$ **c)** $p \rightarrow q$ **d)** $\neg p \rightarrow \neg q$ **e)** $p \rightarrow q$ **f)** $q \wedge \neg p$ **g)** $q \rightarrow p$ **11. a)** $r \wedge \neg p$ **b)** $\neg p \wedge q \wedge r$ **c)** $r \rightarrow (q \leftrightarrow \neg p)$ **d)** $\neg q \wedge \neg p \wedge r$ **e)** $(q \rightarrow (\neg r \wedge \neg p)) \wedge \neg((\neg r \wedge \neg p) \rightarrow q)$ **f)** $(p \wedge r) \rightarrow \neg q$ **13. a)** False **b)** True **c)** True **d)** True **15. a)** Exclusive or: You get only one beverage. **b)** Inclusive or: Long passwords can have any combination of symbols. **c)** Inclusive or: A student with both courses is even more qualified. **d)** Either interpretation possible; a traveler might wish to pay with a mixture of the two currencies, or the store may not allow that. **17. a)** Inclusive or: It is allowable to take discrete mathematics if you have had calculus or computer science, or both. Exclusive or: It is allowable to take discrete mathematics if you have had calculus or computer science, but not if you have had both. Most likely the inclusive or is intended. **b)** Inclusive or: You can take the rebate, or you can get a low-interest loan, or you can get both the rebate and a low-interest loan. Exclusive or: You can take the rebate, or you can get a low-interest loan, but you cannot get both the rebate and a low-interest loan. Most likely the exclusive or is intended. **c)** Inclusive or: You can order two items from column A and none from column B, or three items from column B and none from column A, or five items including two from column A and three from column B. Exclusive or: You can order two itemsfrom column A or three items from column B, but not both. Almost certainly the exclusive or is intended. **d)** Inclusive or: More than 2 feet of snow or windchill below $-100$, or both, will close school. Exclusive or: More than 2 feet of snow or windchill below $-100$, but not both, will close school. Certainly the inclusive or is intended. **19. a)** If the wind blows from the northeast, then it snows. **b)** If it stays warm for a week, then the apple trees will bloom. **c)** If the Pistons win the championship, then they beat the Lakers. **d)** If you get to the top of Long's Peak, then you must have walked 8 miles. **e)** If you are world-famous, then you will get tenure as a professor. **f)** If you drive more than 400 miles, then you will need to buy gasoline. **g)** If your guarantee is good, then you must have bought your CD player less than 90 days ago. **h)** If the water is not too cold, then Jan will go swimming. **21. a)** You buy an ice cream cone if and only if it is hot outside. **b)** You win the contest if and only if you hold the only winning ticket. **c)** You get promoted if and only if you have connections. **d)** Your mind will decay if and only if you watch television. **e)** The train runs late if and only if it is a day I take the train. **23. a)** Converse: "I will ski tomorrow only if it snows today." Contrapositive: "If I do not ski tomorrow, then it will not have snowed today." Inverse: "If it does not snow today, then I will not ski tomorrow." **b)** Converse: "If I come to class, then there will be a quiz." Contrapositive: "If I do not come to class, then there will not be a quiz." Inverse: "If there is not going to be a quiz, then I don't come to class." **c)** Converse: "A positive integer is a prime if it has no divisors other than 1 and itself." Contrapositive: "If a positive integer has a divisor other than 1 and itself, then it is not prime." Inverse: "If a positive integer is not prime, then it has a divisor other than 1 and itself." **25. a)** 2 **b)** 16 **c)** 64 **d)** 16

**27. a)**

| $p$ | $\neg p$ | $p \wedge \neg p$ |
|---|---|---|
| T | F | F |
| F | T | F |

**b)**

| $p$ | $\neg p$ | $p \wedge \neg p$ |
|---|---|---|
| T | F | T |
| F | T | T |

**c)**

| $p$ | $q$ | $\neg q$ | $p \vee \neg q$ | $(p \vee \neg q) \rightarrow q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | T | T | F |
| F | T | F | F | T |
| F | F | T | T | F |

**d)**

| $p$ | $q$ | $p \vee q$ | $p \wedge q$ | $(p \vee q) \rightarrow (p \wedge q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | T | F | F |
| F | T | T | F | F |
| F | F | F | F | T |

**e)**

| $p$ | $q$ | $p \to q$ | $\neg q$ | $\neg p$ | $\neg q \to \neg p$ | $(p \to q) \leftrightarrow (\neg q \to \neg p)$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | T | T | T |

**f)**

| $p$ | $q$ | $p \to q$ | $q \to p$ | $(p \to q) \to (q \to p)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | T | F | F |
| F | F | T | T | T |

**29.** For parts (a), (b), (c), (d), and (f) we have this table.

| $p$ | $q$ | $(p \vee q) \to (p \oplus q)$ | $(p \oplus q) \to (p \wedge q)$ | $(p \vee q) \oplus (p \wedge q)$ | $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$ | $(p \oplus q) \to (p \oplus \neg q)$ |
|---|---|---|---|---|---|---|
| T | T | F | T | F | T | T |
| T | F | T | F | T | T | F |
| F | T | T | F | T | T | F |
| F | F | T | T | F | T | T |

For part (e) we have this table.

| $p$ | $q$ | $r$ | $\neg p$ | $\neg r$ | $p \leftrightarrow q$ | $\neg p \leftrightarrow \neg r$ | $(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T | F |
| T | T | F | F | T | T | F | T |
| T | F | T | F | F | F | T | T |
| T | F | F | F | T | F | F | F |
| F | T | T | T | F | F | F | F |
| F | T | F | T | T | F | T | T |
| F | F | T | T | F | T | F | T |
| F | F | F | T | T | T | T | F |

**31.**

| $p$ | $q$ | $p \to \neg q$ | $\neg p \leftrightarrow q$ | $(p \to q) \vee (\neg p \to q)$ | $(p \to q) \wedge (\neg p \to q)$ | $(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$ | $(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T | T |
| T | F | T | T | T | F | T | T |
| F | T | T | T | T | T | T | T |
| F | F | T | F | T | F | T | T |

**33.**

| $p$ | $q$ | $r$ | $p \to (\neg q \vee r)$ | $\neg p \to (q \to r)$ | $(p \to q) \vee (\neg p \to r)$ | $(p \to q) \wedge (\neg p \to r)$ | $(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$ | $(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | F | T | T | T | T | F |
| T | F | T | T | T | T | F | T | T |
| T | F | F | T | T | T | F | F | F |
| F | T | T | T | T | T | T | F | F |
| F | T | F | T | F | T | F | T | T |
| F | F | T | T | T | T | T | T | F |
| F | F | F | T | T | T | F | T | T |

**35.**

| $p$ | $q$ | $r$ | $s$ | $p \leftrightarrow q$ | $r \leftrightarrow s$ | $(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$ |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | T | F | T | F | F |
| T | T | F | T | T | F | F |
| T | T | F | F | T | T | T |
| T | F | T | T | F | T | F |
| T | F | T | F | F | F | T |
| T | F | F | T | F | F | T |
| T | F | F | F | F | T | F |
| F | T | T | T | F | T | F |
| F | T | T | F | F | F | T |
| F | T | F | T | F | F | T |
| F | T | F | F | F | T | F |
| F | F | T | T | T | T | T |
| F | F | T | F | T | F | F |
| F | F | F | T | T | F | F |
| F | F | F | F | T | T | T |

**37. a)** Bitwise *OR* is 111 1111; bitwise *AND* is 000 0000; bitwise *XOR* is 111 1111. **b)** Bitwise *OR* is 1111 1010; bitwise *AND* is 1010 0000; bitwise *XOR* is 0101 1010. **c)** Bitwise *OR* is 10 0111 1001; bitwise *AND* is 00 0100 0000; bitwise *XOR* is 10 0011 1001. **d)** Bitwise *OR* is 11 1111 1111; bitwise *AND* is 00 0000 0000; bitwise *XOR* is 11 1111 1111. **39.** 0.2, 0.6 **41.** 0.8, 0.6 **43. a)** The 99th statement is true and the rest are false. **b)** Statements 1 through 50 are all true and statements 51 through 100 are all false. **c)** This cannot happen; it is a paradox, showing that these cannot be statements. **45.** "If I were to ask you whether the right branch leads to the ruins, would you answer yes?" **47. a)** $q \to p$ **b)** $q \wedge \neg p$ **c)** $q \to p$ **d)** $\neg q \to \neg p$ **49.** Not consistent **51.** Consistent **53.** NEW **AND** JERSEY **AND** BEACHES, (JERSEY **AND** BEACHES) **NOT** NEW **55.** $A$ is a knight and $B$ is a knave. **57.** $A$ is a knight and $B$ is a knight. **59.** $A$ is a knave and $B$ is a knight. **61.** In order of decreasing salary: Fred, Maggie, Janice **63.** The detective can determine that the butler and cook are lying but cannot determine whether the gardener is telling the truth or whether the handyman is telling the truth. **65.** The Japanese man owns the zebra, and the Norwegian drinks water.

## Section 1.2

**1.** The equivalences follow by showing that the appropriate pairs of columns of this table agree.

| $p$ | $p \wedge T$ | $p \vee F$ | $p \wedge F$ | $p \vee T$ | $p \vee p$ | $p \wedge p$ |
|---|---|---|---|---|---|---|
| T | T | T | F | T | T | T |
| F | F | F | F | T | F | F |

**3. a)**

| $p$ | $q$ | $p \vee q$ | $q \vee p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | T | T |
| F | F | F | F |

**b)**

| $p$ | $q$ | $p \wedge q$ | $q \wedge p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | F |
| F | T | F | F |
| F | F | F | F |

**5.**

| $p$ | $q$ | $r$ | $q \vee r$ | $p \wedge (q \vee r)$ | $p \wedge q$ | $p \wedge r$ | $(p \wedge q) \vee (p \wedge r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | F | T |
| T | F | T | T | T | F | T | T |
| T | F | F | F | F | F | F | F |
| F | T | T | T | F | F | F | F |
| F | T | F | T | F | F | F | F |
| F | F | T | T | F | F | F | F |
| F | F | F | F | F | F | F | F |

**7. a)** Jan is not rich, or Jan is not happy. **b)** Carlos will not bicycle tomorrow, and Carlos will not run tomorrow. **c)** Mei does not walk to class, and Mei does not take the bus to class. **d)** Ibrahim is not smart, or Ibrahim is not hard working.

**9. a)**

| $p$ | $q$ | $p \wedge q$ | $(p \wedge q) \to p$ |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

**b)**

| $p$ | $q$ | $p \vee q$ | $p \to (p \vee q)$ |
|---|---|---|---|
| T | T | T | T |
| T | F | T | T |
| F | T | T | T |
| F | F | F | T |

**c)**

| $p$ | $q$ | $\neg p$ | $p \to q$ | $\neg p \to (p \to q)$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | T |
| F | T | T | T | T |
| F | F | T | T | T |

**d)**

| $p$ | $q$ | $p \wedge q$ | $p \to q$ | $(p \wedge q) \to (p \to q)$ |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | F | T |
| F | T | F | T | T |
| F | F | F | T | T |

e)

| $p$ | $q$ | $p \to q$ | $\neg(p \to q)$ | $\neg(p \to q) \to p$ |
|---|---|---|---|---|
| T | T | T | F | T |
| T | F | F | T | T |
| F | T | T | F | T |
| F | F | T | F | T |

f)

| $p$ | $q$ | $p \to q$ | $\neg(p \to q)$ | $\neg q$ | $\neg(p \to q) \to \neg q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | F | T | T | T |
| F | T | T | F | F | T |
| F | F | T | F | T | T |

**11.** In each case we will show that if the hypothesis is true, then the conclusion is also. **a)** If the hypothesis $p \wedge q$ is true, then by the definition of conjunction, the conclusion $p$ must also be true. **b)** If the hypothesis $p$ is true, by the definition of disjunction, the conclusion $p \vee q$ is also true. **c)** If the hypothesis $\neg p$ is true, that is, if $p$ is false, then the conclusion $p \to q$ is true. **d)** If the hypothesis $p \wedge q$ is true, then both $p$ and $q$ are true, so the conclusion $p \to q$ is also true. **e)** If the hypothesis $\neg(p \to q)$ is true, then $p \to q$ is false, so the conclusion $p$ is true (and $q$ is false). **f)** If the hypothesis $\neg(p \to q)$ is true, then $p \to q$ is false, so $p$ is true and $q$ is false. Hence, the conclusion $\neg q$ is true. **13.** That the fourth column of the truth table shown is identical to the first column proves part **(a)**, and that the sixth column is identical to the first column proves part **(b)**.

| $p$ | $q$ | $p \wedge q$ | $p \vee (p \wedge q)$ | $p \vee q$ | $p \wedge (p \vee q)$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | T | T |
| F | T | F | F | T | F |
| F | F | F | F | F | F |

**15.** It is a tautology. **17.** Each of these is true precisely when $p$ and $q$ have opposite truth values. **19.** The proposition $\neg p \leftrightarrow q$ is true when $\neg p$ and $q$ have the same truth values, which means that $p$ and $q$ have different truth values. Similarly, $p \leftrightarrow \neg q$ is true in exactly the same cases. Therefore, these two expressions are logically equivalent. **21.** The proposition $\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false, which means that $p$ and $q$ have different truth values. Because this is precisely when $\neg p \leftrightarrow q$ is true, the two expressions are logically equivalent. **23.** For $(p \to r) \wedge (q \to r)$ to be false, one of the two conditional statements must be false, which happens exactly when $r$ is false and at least one of $p$ and $q$ is true. But these are precisely the cases in which $p \vee q$ is true and $r$ is false, which is precisely when $(p \vee q) \to r$ is false. Because the two propositions are false in exactly the same situations, they are logically equivalent. **25.** For $(p \to r) \vee (q \to r)$ to be false, both of the two conditional statements must be false, which happens exactly when $r$ is false and both $p$ and $q$ are true. But this is precisely the case in which $p \wedge q$ is true and $r$ is false, which is precisely when $(p \wedge q) \to r$ is false. Because the two propositions are false in exactly the same situations, they are logically equivalent. **27.** This fact was observed in Section 1 when the biconditional was first defined. Each of these is true precisely when $p$ and $q$ have the same truth values. **29.** The last column is all Ts.

| $p$ | $q$ | $r$ | $p \to q$ | $q \to r$ | $(p \to q) \wedge (q \to r)$ | $p \to r$ | $(p \to q) \wedge (q \to r) \to (p \to r)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | T | F | T | T |
| T | F | F | F | T | F | F | T |
| F | T | T | T | T | T | T | T |
| F | T | F | T | F | F | T | T |
| F | F | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T |

**31.** These are not logically equivalent because when $p$, $q$, and $r$ are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true. **33.** Many answers are possible. If we let $r$ be true and $p$, $q$, and $s$ be false, then $(p \to q) \to (r \to s)$ will be false, but $(p \to r) \to (q \to s)$ will be true. **35. a)** $p \vee \neg q \vee \neg r$ **b)** $(p \vee q \vee r) \wedge s$ **c)** $(p \wedge \mathbf{T}) \vee (q \wedge \mathbf{F})$ **37.** If we take duals twice, every $\vee$ changes to an $\wedge$ and then back to an $\vee$, every $\wedge$ changes to an $\vee$ and then back to an $\wedge$, every **T** changes to an **F** and then back to a **T**, every **F** changes to a **T** and then back to an **F**. Hence, $(s^*)^* = s$. **39.** Let $p$ and $q$ be equivalent compound propositions involving only the operators $\wedge$, $\vee$, and $\neg$, and **T** and **F**. Note that $\neg p$ and $\neg q$ are also equivalent. Use De Morgan's laws as many times as necessary to push negations in as far as possible within these compound propositions, changing $\vee$s to $\wedge$s, and vice versa, and changing **T**s to **F**s, and vice versa. This shows that $\neg p$ and $\neg q$ are the same as $p^*$ and $q^*$ except that each atomic proposition $p_i$ within them is replaced by its negation. From this we can conclude that $p^*$ and $q^*$ are equivalent because $\neg p$ and $\neg q$ are. **41.** $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$ **43.** Given a compound proposition $p$, form its truth table and then write down a proposition $q$ in disjunctive normal form that is logically equivalent to $p$. Because $q$ involves only $\neg$, $\wedge$, and $\vee$, this shows that these three operators form a functionally complete set. **45.** By Exercise 43, given a compound proposition $p$, we can write down a proposition $q$ that is logically equivalent to $p$ and involves only $\neg$, $\wedge$, and $\vee$. By De Morgan's law we can eliminate all the $\wedge$'s by replacing each occurrence of $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ with $\neg(\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$. **47.** $\neg(p \wedge q)$ is true when either $p$ or $q$, or both, are false, and is false when both $p$ and $q$ are true. Because this was the definition of $p \mid q$, the two compound propositions are logically equivalent. **49.** $\neg(p \vee q)$ is true when both $p$ and $q$ are false, and is false otherwise. Because this was the definition of $p \downarrow q$, the two are logically equivalent. **51.** $((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q)$ **53.** This follows immediately from the truth table or definition of $p \mid q$. **55.** 16 **57.** If the database is open, then either the system is in its initial state or the monitor is put in a closed state. **59.** All nine **61.** To determine whether $c$ is a tautology apply an algorithm for satisfiability to $\neg c$. If the algorithm says that $\neg c$ is satisfiable, then we report that $c$ is not a tautology, and if the algorithm says that $\neg c$ is not satisfiable, then we report that $c$ is a tautology.

## Section 1.3

**1. a)** T **b)** T **c)** F **3. a)** T **b)** F **c)** F **d)** F **5. a)** There is a student who spends more than 5 hours every weekday in class. **b)** Every student spends more than 5 hours every weekday in class. **c)** There is a student who does not spend more than 5 hours every weekday in class. **d)** No student spends more than 5 hours every weekday in class. **7. a)** Every comedian is funny. **b)** Every person is a funny comedian. **c)** There exists a person such that if she or he is a comedian, then she or he is funny. **d)** Some comedians are funny. **9. a)** $\exists x(P(x) \wedge Q(x))$ **b)** $\exists x(P(x) \wedge \neg Q(x))$ **c)** $\forall x(P(x) \vee Q(x))$ **d)** $\forall x \neg(P(x) \vee Q(x))$ **11. a)** T **b)** T **c)** F **d)** F **e)** T **f)** F **13. a)** True **b)** True **c)** True **d)** True **15. a)** True **b)** False **c)** True **d)** False **17. a)** $P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4)$ **b)** $P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4)$ **c)** $\neg P(0) \vee \neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4)$ **d)** $\neg P(0) \wedge \neg P(1) \wedge \neg P(2) \wedge \neg P(3) \wedge \neg P(4)$ **e)** $\neg(P(0) \vee P(1) \vee P(2) \vee P(3) \vee P(4))$ **f)** $\neg(P(0) \wedge P(1) \wedge P(2) \wedge P(3) \wedge P(4))$ **19. a)** $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$ **b)** $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$ **c)** $\neg(P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5))$ **d)** $\neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5))$ **e)** $(P(1) \wedge P(2) \wedge P(4) \wedge P(5)) \vee (\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$ **21.** Many answers are possible. **a)** All students in your discrete mathematics class; all students in the world **b)** All United States senators; all college football players **c)** George W. Bush and Jeb Bush; all politicians in the United States **d)** Bill Clinton and George W. Bush; all politicians in the United States **23.** Let $C(x)$ be the propositional function "$x$ is in your class." **a)** $\exists x H(x)$ and $\exists x(C(x) \wedge H(x))$, where $H(x)$ is "$x$ can speak Hindi" **b)** $\forall x F(x)$ and $\forall x(C(x) \rightarrow F(x))$, where $F(x)$ is "$x$ is friendly" **c)** $\exists x \neg B(x)$ and $\exists x(C(x) \wedge \neg B(x))$, where $B(x)$ is "$x$ was born in California" **d)** $\exists x M(x)$ and $\exists x(C(x) \wedge M(x))$, where $M(x)$ is "$x$ has been in a movie" **e)** $\forall x \neg L(x)$ and $\forall x(C(x) \rightarrow \neg L(x))$, where $L(x)$ is "$x$ has taken a course in logic programming" **25.** Let $P(x)$ be "$x$ is perfect"; let $F(x)$ be "$x$ is your friend"; and let the domain be all people. **a)** $\forall x \neg P(x)$ **b)** $\neg \forall x\, P(x)$ **c)** $\forall x(F(x) \rightarrow P(x))$ **d)** $\exists x(F(x) \wedge P(x))$ **e)** $\forall x(F(x) \wedge P(x))$ or $(\forall x\, F(x)) \wedge (\forall x\, P(x))$ **f)** $(\neg \forall x\, F(x)) \vee (\exists x \neg P(x))$ **27.** Let $Y(x)$ be the propositional function that $x$ is in your school or class, as appropriate. **a)** If we let $V(x)$ be "$x$ has lived in Vietnam," then we have $\exists x V(x)$ if the domain is just your schoolmates, or $\exists x(Y(x) \wedge V(x))$ if the domain is all people. If we let $D(x, y)$ mean that person $x$ has lived in country $y$, then we can rewrite this last one as $\exists x(Y(x) \wedge D(x, \text{Vietnam}))$. **b)** If we let $H(x)$ be "$x$ can speak Hindi," then we have $\exists x \neg H(x)$ if the domain is just your schoolmates, or $\exists x(Y(x) \wedge \neg H(x))$ if the domain is all people. If we let $S(x, y)$ mean that person $x$ can speak language $y$, then we can rewrite this last one as $\exists x(Y(x) \wedge \neg S(x, \text{Hindi}))$. **c)** If we let $J(x)$, $P(x)$, and $C(x)$ be the propositional functions asserting $x$'s knowledge of Java, Prolog, and C++, respectively, then we have $\exists x(J(x) \wedge P(x) \wedge C(x))$ if the domain is just your schoolmates, or $\exists x(Y(x) \wedge J(x) \wedge P(x) \wedge C(x))$ if the domain is all people. If we let $K(x, y)$ mean that person $x$ knows programming language $y$, then we can rewrite this last one as $\exists x(Y(x) \wedge K(x, \text{Java}) \wedge K(x, \text{Prolog}) \wedge K(x, \text{C++}))$. **d)** If we let $T(x)$ be "$x$ enjoys Thai food," then we have $\forall x\, T(x)$ if the domain is just your classmates, or $\forall x(Y(x) \rightarrow T(x))$ if the domain is all people. If we let $E(x, y)$ mean that person $x$ enjoys food of type $y$, then we can rewrite this last one as $\forall x(Y(x) \rightarrow E(x, \text{Thai}))$. **e)** If we let $H(x)$ be "$x$ plays hockey," then we have $\exists x \neg H(x)$ if the domain is just your classmates, or $\exists x(Y(x) \wedge \neg H(x))$ if the domain is all people. If we let $P(x, y)$ mean that person $x$ plays game $y$, then we can rewrite this last one as $\exists x(Y(x) \wedge \neg P(x, \text{hockey}))$. **29.** Let $T(x)$ mean that $x$ is a tautology and $C(x)$ mean that $x$ is a contradiction. **a)** $\exists x\, T(x)$ **b)** $\forall x(C(x) \rightarrow T(\neg x))$ **c)** $\exists x \exists y(\neg T(x) \wedge \neg C(x) \wedge \neg T(y) \wedge \neg C(y) \wedge T(x \vee y))$ **d)** $\forall x \forall y((T(x) \wedge T(y)) \rightarrow T(x \wedge y))$ **31. a)** $Q(0,0,0) \wedge Q(0,1,0)$ **b)** $Q(0,1,1) \vee Q(1,1,1) \vee Q(2,1,1)$ **c)** $\neg Q(0,0,0) \vee \neg Q(0,0,1)$ **d)** $\neg Q(0,0,1) \vee \neg Q(1,0,1) \vee \neg Q(2,0,1)$ **33. a)** Let $T(x)$ be the predicate that $x$ can learn new tricks, and let the domain be old dogs. Original is $\exists x\, T(x)$. Negation is $\forall x \neg T(x)$: "No old dogs can learn new tricks." **b)** Let $C(x)$ be the predicate that $x$ knows calculus, and let the domain be rabbits. Original is $\neg \exists x\, C(x)$. Negation is $\exists x\, C(x)$: "There is a rabbit that knows calculus." **c)** Let $F(x)$ be the predicate that $x$ can fly, and let the domain be birds. Original is $\forall x\, F(x)$. Negation is $\exists x \neg F(x)$: "There is a bird who cannot fly." **d)** Let $T(x)$ be the predicate that $x$ can talk, and let the domain be dogs. Original is $\neg \exists x\, T(x)$. Negation is $\exists x\, T(x)$: "There is a dog that talks." **e)** Let $F(x)$ and $R(x)$ be the predicates that $x$ knows French and knows Russian, respectively, and let the domain be people in this class. Original is $\neg \exists x(F(x) \wedge R(x))$. Negation is $\exists x(F(x) \wedge R(x))$: "There is someone in this class who knows French and Russian." **35. a)** There is no counterexample. **b)** $x = 0$ **c)** $x = 2$ **37. a)** $\forall x((F(x, 25{,}000) \vee S(x, 25)) \rightarrow E(x))$, where $E(x)$ is "Person $x$ qualifies as an elite flyer in a given year," $F(x, y)$ is "Person $x$ flies more than $y$ miles in a given year," and $S(x, y)$ is "Person $x$ takes more than $y$ flights in a given year" **b)** $\forall x(((M(x) \wedge T(x, 3)) \vee (\neg M(x) \wedge T(x, 3.5))) \rightarrow Q(x))$, where $Q(x)$ is "Person $x$ qualifies for the marathon," $M(x)$ is "Person $x$ is a man," and $T(x, y)$ is "Person $x$ has run the marathon in less than $y$ hours" **c)** $M \rightarrow ((H(60) \vee (H(45) \wedge T)) \wedge \forall y\, G(\text{B}, y))$, where $M$ is the proposition "The student received a masters degree," $H(x)$ is "The student took at least $x$ course hours," $T$ is the proposition "The student wrote a thesis," and $G(x, y)$ is "The person got grade $x$ or higher in course $y$" **d)** $\exists x\, ((T(x, 21) \wedge G(x, 4.0))$, where $T(x, y)$ is "Person $x$ took more than $y$ credit hours" and $G(x, p)$ is "Person $x$ earned grade point average $p$" (we assume that we are talking about one given semester)

**39. a)** If there is a printer that is both out of service and busy, then some job has been lost. **b)** If every printer is busy, then there is a job in the queue. **c)** If there is a job that is both queued and lost, then some printer is out of service. **d)** If every printer is busy and every job is queued, then some job is lost. **41. a)** $(\exists x\, F(x, 10)) \to \exists x\, S(x)$, where $F(x, y)$ is "Disk $x$ has more than $y$ kilobytes of free space," and $S(x)$ is "Mail message $x$ can be saved" **b)** $(\exists x\, A(x)) \to \forall x(Q(x) \to T(x))$, where $A(x)$ is "Alert $x$ is active," $Q(x)$ is "Message $x$ is queued," and $T(x)$ is "Message $x$ is transmitted" **c)** $\forall x((x \neq \text{main console}) \to T(x))$, where $T(x)$ is "The diagnostic monitor tracks the status of system $x$" **d)** $\forall x(\neg L(x) \to B(x))$, where $L(x)$ is "The host of the conference call put participant $x$ on a special list" and $B(x)$ is "Participant $x$ was billed" **43.** They are not equivalent. Let $P(x)$ be any propositional function that is sometimes true and sometimes false, and let $Q(x)$ be any propositional function that is always false. Then $\forall x(P(x) \to Q(x))$ is false but $\forall x P(x) \to \forall x Q(x)$ is true. **45.** Both statements are true precisely when at least one of $P(x)$ and $Q(x)$ is true for at least one value of $x$ in the domain. **47. a)** If $A$ is true, then both sides are logically equivalent to $\forall x P(x)$. If $A$ is false, the left-hand side is clearly false. Furthermore, for every $x$, $P(x) \wedge A$ is false, so the right-hand side is false. Hence, the two sides are logically equivalent. **b)** If $A$ is true, then both sides are logically equivalent to $\exists x\ P(x)$. If $A$ is false, the left-hand side is clearly false. Furthermore, for every $x$, $P(x) \wedge A$ is false, so $\exists x(P(x) \wedge A)$ is false. Hence, the two sides are logically equivalent. **49.** We can establish these equivalences by arguing that one side is true if and only if the other side is true. **a)** Suppose that $A$ is true. Then for each $x$, $P(x) \to A$ is true; therefore the left-hand side is always true in this case. By similar reasoning the right-hand side is always true in this case. Therefore, the two propositions are logically equivalent when $A$ is true. On the other hand, suppose that $A$ is false. There are two subcases. If $P(x)$ is false for every $x$, then $P(x) \to A$ is vacuously true, so the left-hand side is vacuously true. The same reasoning shows that the right-hand side is also true, because in this subcase $\exists x P(x)$ is false. For the second subcase, suppose that $P(x)$ is true for some $x$. Then for that $x$, $P(x) \to A$ is false, so the left-hand side is false. The right-hand side is also false, because in this subcase $\exists x P(x)$ is true but $A$ is false. Thus in all cases, the two propositions have the same truth value. **b)** If $A$ is true, then both sides are trivially true, because the conditional statements have true conclusions. If $A$ is false, then there are two subcases. If $P(x)$ is false for some $x$, then $P(x) \to A$ is vacuously true for that $x$, so the left-hand side is true. The same reasoning shows that the right-hand side is true, because in this subcase $\forall x P(x)$ is false. For the second subcase, suppose that $P(x)$ is true for every $x$. Then for every $x$, $P(x) \to A$ is false, so the left-hand side is false (there is no $x$ making the conditional statement true). The right-hand side is also false, because it is a conditional statement with a true hypothesis and a false conclusion. Thus in all cases, the two propositions have the same truth value. **51.** To show these are not logically equivalent, let $P(x)$ be the statement "$x$ is positive," and let $Q(x)$ be the statement "$x$ is negative" with domain the set of integers. Then $\exists x\ P(x) \wedge \exists x\ Q(x)$ is true, but $\exists x(P(x) \wedge Q(x))$ is false. **53. a)** True **b)** False, unless the domain consists of just one element **c)** True **55. a)** Yes **b)** No **c)** juana, kiko **d)** math273, cs301 **e)** juana, kiko **57.** `sibling(X,Y) :- mother(M,X), mother(M,Y), father(F,X), father(F,Y)` **59. a)** $\forall x(P(x) \to \neg Q(x))$ **b)** $\forall x(Q(x) \to R(x))$ **c)** $\forall x(P(x) \to \neg R(x))$ **d)** The conclusion does not follow. There may be vain professors, because the premises do not rule out the possibility that there are other vain people besides ignorant ones. **61. a)** $\forall x(P(x) \to \neg Q(x))$ **b)** $\forall x(R(x) \to \neg S(x))$ **c)** $\forall x(\neg Q(x) \to S(x))$ **d)** $\forall x(P(x) \to \neg R(x))$ **e)** The conclusion follows. Suppose $x$ is a baby. Then by the first premise, $x$ is illogical, so by the third premise, $x$ is despised. The second premise says that if $x$ could manage a crocodile, then $x$ would not be despised. Therefore, $x$ cannot manage a crocodile.

## Section 1.4

**1. a)** For every real number $x$ there exists a real number $y$ such that $x$ is less than $y$. **b)** For every real number $x$ and real number $y$, if $x$ and $y$ are both nonnegative, then their product is nonnegative. **c)** For every real number $x$ and real number $y$, there exists a real number $z$ such that $xy = z$. **3. a)** There is some student in your class who has sent a message to some student in your class. **b)** There is some student in your class who has sent a message to every student in your class. **c)** Every student in your class has sent a message to at least one student in your class. **d)** There is a student in your class who has been sent a message by every student in your class. **e)** Every student in your class has been sent a message from at least one student in your class. **f)** Every student in the class has sent a message to every student in the class. **5. a)** Sarah Smith has visited www.att.com. **b)** At least one person has visited www.imdb.org. **c)** Jose Orez has visited at least one website. **d)** There is a website that both Ashok Puri and Cindy Yoon have visited. **e)** There is a person besides David Belcher who has visited all the websites that David Belcher has visited. **f)** There are two different people who have visited exactly the same websites. **7. a)** Abdallah Hussein does not like Japanese cuisine. **b)** Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine. **c)** There is some cuisine that either Monique Arsenault or Jay Johnson likes. **d)** For every pair of distinct students at your school, there is some cuisine that at least one them does not like. **e)** There are two students at your school who like exactly the same set of cuisines. **f)** For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it). **9. a)** $\forall x L(x, \text{Jerry})$ **b)** $\forall x \exists y L(x, y)$ **c)** $\exists y \forall x L(x, y)$ **d)** $\forall x \exists y \neg L(x, y)$ **e)** $\exists x \neg L(\text{Lydia}, x)$ **f)** $\exists x \forall y \neg L(y, x)$ **g)** $\exists x(\forall y L(y, x) \wedge \forall z((\forall w L(w, z)) \to z = x))$ **h)** $\exists x \exists y(x \neq y \wedge L(\text{Lynn}, x) \wedge L(\text{Lynn}, y) \wedge \forall z(L(\text{Lynn}, z) \to (z = x \vee z = y)))$ **i)** $\forall x L(x, x)$ **j)** $\exists x \forall y (L(x,y) \leftrightarrow x = y)$ **11. a)** $A(\text{Lois, Professor Michaels})$

**b)** $\forall x(S(x) \to A(x, \text{Professor Gross}))$ **c)** $\forall x(F(x) \to (A(x, \text{Professor Miller}) \vee A(\text{Professor Miller}, x)))$ **d)** $\exists x(S(x) \wedge \forall y(F(y) \to \neg A(x, y)))$ **e)** $\exists x(F(x) \wedge \forall y(S(y) \to \neg A(y,x)))$ **f)** $\forall y(F(y) \to \exists x(S(x) \vee A(x,y)))$ **g)** $\exists x(F(x) \wedge \forall y((F(y) \wedge (y \neq x)) \to A(x,y)))$ **h)** $\exists x(S(x) \wedge \forall y(F(y) \to \neg A(y, x)))$ **13. a)** $\neg M$ (Chou, Koko) **b)** $\neg M$(Arlene, Sarah) $\wedge \neg T$(Arlene, Sarah) **c)** $\neg M$ (Deborah, Jose) **d)** $\forall x M(x, \text{Ken})$ **e)** $\forall x \neg T(x, \text{Nina})$ **f)** $\forall x(T\text{-}x, \text{Avi}) \vee M(x, \text{Avi}))$ **g)** $\exists x \forall y(y \neq x \to M(x, y))$ **h)** $\exists x \forall y(y \neq x \to (M(x, y) \vee T(x, y)))$ **i)** $\exists x \exists y(x \neq y \wedge M(x, y) \wedge M(y, x))$ **j)** $\exists x M(x, x)$ **k)** $\exists x \forall y(x \neq y \to (\neg M(x, y) \wedge \neg T(y, x)))$ **l)** $\forall x(\exists y(x \neq y \wedge (M(y, x) \vee T(y, x))))$ **m)** $\exists x \exists y(x \neq y \wedge M(x, y) \wedge T(y, x))$ **n)** $\exists x \exists y(x \neq y \wedge \forall z((z \neq x \wedge z \neq y) \to (M(x, z) \vee M(y, z) \vee T(x, z) \vee T(y, z))))$ **15. a)** $\forall x P(x)$, where $P(x)$ is "$x$ needs a course in discrete mathematics" and the domain consists of all computer science students **b)** $\exists x P(x)$, where $P(x)$ is "$x$ owns a personal computer" and the domain consists of all students in this class **c)** $\forall x \exists y P(x, y)$, where $P(x, y)$ is "$x$ has taken $y$," the domain for $x$ consists of all students in this class, and the domain for $y$ consists of all computer science classes **d)** $\exists x \exists y P(x, y)$, where $P(x, y)$ and domains are the same as in part (c) **e)** $\forall x \forall y P(x, y)$, where $P(x, y)$ is "$x$ has been in $y$," the domain for $x$ consists of all students in this class, and the domain for $y$ consists of all buildings on campus **f)** $\exists x \exists y \forall z(P(z, y) \to Q(x, z))$, where $P(z, y)$ is "$z$ is in $y$" and $Q(x, z)$ is "$x$ has been in $z$"; the domain for $x$ consists of all students in the class, the domain for $y$ consists of all buildings on campus, and the domain of $z$ consists of all rooms. **g)** $\forall x \forall y \exists z(P(z, y) \wedge Q(x, z))$, with same environment as in part (f) **17. a)** $\forall u \exists m(A(u, m) \wedge \forall n(n \neq m \to \neg A(u, n)))$, where $A(u, m)$ means that user $u$ has access to mailbox $m$ **b)** $\exists p \forall e(H(e) \wedge S(p, \text{running})) \to S$ (kernel, working correctly), where $H(e)$ means that error condition $e$ is in effect and $S(x, y)$ means that the status of $x$ is $y$ **c)** $\forall u \forall s(E(s, .\text{edu}) \to A(u, s))$, where $E(s, x)$ means that website $s$ has extension $x$, and $A(u, s)$ means that user $u$ can access website $s$ **d)** $\exists x \exists y(x \neq y \wedge \forall z((\forall s\, M(z,s)) \leftrightarrow (z = x \vee z = y)))$, where $M(a, b)$ means that system $a$ monitors remote server $b$ **19. a)** $\forall x \forall y((x < 0) \wedge (y < 0) \to (x + y < 0))$ **b)** $\neg \forall x \forall y((x > 0) \wedge (y > 0) \to (x - y > 0))$ **c)** $\forall x \forall y\,(x^2 + y^2 \geq (x + y)^2)$ **d)** $\forall x \forall y\,(|xy| = |x||y|)$ **21.** $\forall x \exists a \exists b \exists c \exists d((x > 0) \to x = a^2 + b^2 + c^2 + d^2)$, where the domain consists of all integers **23. a)** $\forall x\, \forall y\,((x < 0) \wedge (y < 0) \to (xy > 0))$ **b)** $\forall x(x - x = 0)$ **c)** $\forall x \exists a \exists b(a \neq b \wedge \forall c(c^2 = x \leftrightarrow (c = a \vee c = b)))$ **d)** $\forall x((x < 0) \to \neg \exists y(x = y^2))$ **25. a)** There is a multiplicative identity for the real numbers. **b)** The product of two negative real numbers is always a positive real number. **c)** There exist real numbers $x$ and $y$ such that $x^2$ exceeds $y$ but $x$ is less than $y$. **d)** The real numbers are closed under the operation of addition. **27. a)** True **b)** True **c)** True **d)** True **e)** True **f)** False **g)** False **h)** True **i)** False **29. a)** $P(1,1) \wedge P(1,2) \wedge P(1,3) \wedge P(2,1) \wedge P(2,2) \wedge P(2, 3) \wedge P(3, 1) \wedge P(3, 2) \wedge P(3, 3)$ **b)** $P(1, 1) \vee P(1, 2) \vee P(1, 3) \vee P(2, 1) \vee P(2, 2) \vee P(2, 3) \vee P(3,1) \vee P(3, 2) \vee P(3, 3)$ **c)** $(P(1, 1) \wedge P(1, 2) \wedge P(1, 3)) \vee (P(2, 1) \wedge P(2, 2) \wedge P(2, 3)) \vee (P(3, 1) \wedge P(3, 2) \wedge P(3, 3))$ **d)** $(P(1, 1) \vee P(2, 1) \vee P(3, 1)) \wedge (P(1, 2) \vee P(2, 2) \vee P(3, 2)) \wedge (P(1, 3) \vee P(2, 3) \vee P(3, 3))$ **31. a)** $\exists x \forall y \exists z \neg T(x, y, z)$ **b)** $\exists x \forall y \neg P(x, y) \wedge \exists x \forall y \neg Q(x, y)$ **c)** $\exists x \forall y(\neg P(x, y) \vee \forall z \neg R(x, y, z))$ **d)** $\exists x \forall y(P(x, y) \wedge \neg Q(x, y))$ **33. a)** $\exists x \exists y \neg P(x, y)$ **b)** $\exists y \forall x \neg P(x, y)$ **c)** $\exists y \exists x(\neg P(x, y) \wedge \neg Q(x, y))$ **d)** $(\forall x \forall y P(x, y)) \vee (\exists x \exists y \neg Q(x, y))$ **e)** $\exists x(\forall y \exists z \neg P(x,y,z) \vee \forall z \exists y \neg P(x, y, z))$ **35.** Any domain with four or more members makes the statement true; any domain with three or fewer members makes the statement false. **37. a)** There is someone in this class such that for every two different math courses, these are not the two and only two math courses this person has taken. **b)** Every person has either visited Libya or has not visited a country other than Libya. **c)** Someone has climbed every mountain in the Himalayas. **d)** There is someone who has neither been in a movie with Kevin Bacon nor has been in a movie with someone who has been in a movie with Kevin Bacon. **39. a)** $x = 2,\ y = -2$ **b)** $x = -4$ **c)** $x = 17, y = -1$ **41.** $\forall x\, \forall y\, \forall z((x \cdot y) \cdot z = x \cdot (y \cdot z))$ **43.** $\forall m \forall b(m \neq 0 \to \exists x(mx + b = 0 \wedge \forall w(mw + b = 0 \to w = x)))$ **45. a)** True **b)** False **c)** True **47.** $\neg(\exists x \forall y P(x, y)) \leftrightarrow \forall x(\neg \forall y P(x, y)) \leftrightarrow \forall x \exists y \neg P(x,y)$ **49. a)** Suppose that $\forall x P(x) \wedge \exists x Q(x)$ is true. Then $P(x)$ is true for all $x$ and there is an element $y$ for which $Q(y)$ is true. Because $P(x) \wedge Q(y)$ is true for all $x$ and there is a $y$ for which $Q(y)$ is true, $\forall x \exists y(P(x) \wedge Q(y))$ is true. Conversely, suppose that the second proposition is true. Let $x$ be an element in the domain. There is a $y$ such that $Q(y)$ is true, so $\exists x Q(x)$ is true. Because $\forall x P(x)$ is also true, it follows that the first proposition is true. **b)** Suppose that $\forall x P(x) \vee \exists x Q(x)$ is true. Then either $P(x)$ is true for all $x$, or there exists a $y$ for which $Q(y)$ is true. In the former case, $P(x) \vee Q(y)$ is true for all $x$, so $\forall x \exists y(P(x) \vee Q(y))$ is true. In the latter case, $Q(y)$ is true for a particular $y$, so $P(x) \vee Q(y)$ is true for all $x$ and consequently $\forall x \exists y(P(x) \vee Q(y))$ is true. Conversely, suppose that the second proposition is true. If $P(x)$ is true for all $x$, then the first proposition is true. If not, $P(x)$ is false for some $x$, and for this $x$ there must be a $y$ such that $P(x) \vee Q(y)$ is true. Hence, $Q(y)$ must be true, so $\exists y Q(y)$ is true. It follows that the first proposition must hold. **51.** We will show how an expression can be put into prenex normal form (PNF) if subexpressions in it can be put into PNF. Then, working from the inside out, any expression can be put in PNF. (To formalize the argument, it is necessary to use the method of structural induction that will be discussed in Section 4.3.) By Exercise 45 of Section 1.2, we can assume that the proposition uses only $\vee$ and $\neg$ as logical connectives. Now note that any proposition with no quantifiers is already in PNF. (This is the basis case of the argument.) Now suppose that the proposition is of the form $Qx P(x)$, where $Q$ is a quantifier. Because $P(x)$ is a shorter expression than the original proposition, we can put it into PNF. Then $Qx$ followed by this PNF is again in PNF and is equivalent to the original proposition. Next, suppose that the proposition is of the form $\neg P$. If $P$ is already in PNF, we slide the negation sign past all the quantifiers using the equivalences in Table 2 in Section 1.3. Finally, assume that proposition is of the form $P \vee Q$, where each of $P$ and $Q$ is in PNF. If only one of $P$ and $Q$ has quantifiers, then we can use Exercise 46 in Section 1.3 to bring the quantifier in front of both. If both $P$ and $Q$ have quantifiers, we can use Exercise 45 in Section 1.3, Exercise 48, or part (b) of Exercise 49 to rewrite $P \vee Q$ with two quantifiers preceding the disjunction of a proposition of the form $R \vee S$, and then put $R \vee S$ into PNF.

## Section 1.5

**1.** Modus ponens; valid; the conclusion is true, because the hypotheses are true. **3. a)** Addition **b)** Simplification **c)** Modus ponens **d)** Modus tollens **e)** Hypothetical syllogism **5.** Let $w$ be "Randy works hard," let $d$ be "Randy is a dull boy," and let $j$ be "Randy will get the job." The hypotheses are $w$, $w \rightarrow d$, and $d \rightarrow \neg j$. Using modus ponens and the first two hypotheses, $d$ follows. Using modus ponens and the last hypothesis, $\neg j$, which is the desired conclusion, "Randy will not get the job," follows. **7.** Universal instantiation is used to conclude that "If Socrates is a man, then Socrates is mortal." Modus ponens is then used to conclude that Socrates is mortal. **9. a)** Valid conclusions are "I did not take Tuesday off," "I took Thursday off," "It rained on Thursday." **b)** "I did not eat spicy foods and it did not thunder" is a valid conclusion. **c)** "I am clever" is a valid conclusion. **d)** "Ralph is not a CS major" is a valid conclusion. **e)** "That you buy lots of stuff is good for the U.S. and is good for you" is a valid conclusion. **f)** "Mice gnaw their food" and "Rabbits are not rodents" are valid conclusions. **11.** Suppose that $p_1, p_2, \ldots, p_n$ are true. We want to establish that $q \rightarrow r$ is true. If $q$ is false, then we are done, vacuously. Otherwise, $q$ is true, so by the validity of the given argument form (that whenever $p_1, p_2, \ldots, p_n, q$ are true, then $r$ must be true), we know that $r$ is true. **13. a)** Let $c(x)$ be "$x$ is in this class," $j(x)$ be "$x$ knows how to write programs in JAVA," and $h(x)$ be "$x$ can get a high-paying job." The premises are $c(\text{Doug})$, $j(\text{Doug})$, $\forall x(j(x) \rightarrow h(x))$. Using universal instantiation and the last premise, $j(\text{Doug}) \rightarrow h(\text{Doug})$ follows. Applying modus ponens to this conclusion and the second premise, $h(\text{Doug})$ follows. Using conjunction and the first premise, $c(\text{Doug}) \wedge h(\text{Doug})$ follows. Finally, using existential generalization, the desired conclusion, $\exists x(c(x) \wedge h(x))$ follows. **b)** Let $c(x)$ be "$x$ is in this class," $w(x)$ be "$x$ enjoys whale watching," and $p(x)$ be "$x$ cares about ocean pollution." The premises are $\exists x(c(x) \wedge w(x))$ and $\forall x(w(x) \rightarrow p(x))$. From the first premise, $c(y) \wedge w(y)$ for a particular person $y$. Using simplification, $w(y)$ follows. Using the second premise and universal instantiation, $w(y) \rightarrow p(y)$ follows. Using modus ponens, $p(y)$ follows, and by conjunction, $c(y) \wedge p(y)$ follows. Finally, by existential generalization, the desired conclusion, $\exists x(c(x) \wedge p(x))$, follows. **c)** Let $c(x)$ be "$x$ is in this class," $p(x)$ be "$x$ owns a PC," and $w(x)$ be "$x$ can use a word-processing program." The premises are $c(\text{Zeke})$, $\forall x(c(x) \rightarrow p(x))$, and $\forall x(p(x) \rightarrow w(x))$. Using the second premise and universal instantiation, $c(\text{Zeke}) \rightarrow p(\text{Zeke})$ follows. Using the first premise and modus ponens, $p(\text{Zeke})$ follows. Using the third premise and universal instantiation, $p(\text{Zeke}) \rightarrow w(\text{Zeke})$ follows. Finally, using modus ponens, $w(\text{Zeke})$, the desired conclusion, follows. **d)** Let $j(x)$ be "$x$ is in New Jersey," $f(x)$ be "$x$ lives within 50 miles of the ocean," and $s(x)$ be "$x$ has seen the ocean." The premises are $\forall x(j(x) \rightarrow f(x))$ and $\exists x(j(x) \wedge \neg s(x))$. The second hypothesis and existential instantiation imply that $j(y) \wedge \neg s(y)$ for a particular person $y$. By simplification, $j(y)$ for this person $y$. Using universal instantiation and the first premise, $j(y) \rightarrow f(y)$, and by modus ponens, $f(y)$ follows. By simplification, $\neg s(y)$ follows from $j(y) \wedge \neg s(y)$. So $f(y) \wedge \neg s(y)$ follows by conjunction. Finally, the desired conclusion, $\exists x(f(x) \wedge \neg s(x))$, follows by existential generalization. **15. a)** Correct, using universal instantiation and modus ponens . **b)** Invalid; fallacy of affirming the conclusion **c)** Invalid; fallacy of denying the hypothesis **d)** Correct, using universal instantiation and modus tollens **17.** We know that *some* $x$ exists that makes $H(x)$ true, but we cannot conclude that Lola is one such $x$. **19. a)** Fallacy of affirming the conclusion **b)** Fallacy of begging the question **c)** Valid argument using modus tollens **d)** Fallacy of denying the hypothesis **21.** By the second premise, there is some lion that does not drink coffee. Let Leo be such a creature. By simplification we know that Leo is a lion. By modus ponens we know from the first premise that Leo is fierce. Hence, Leo is fierce and does not drink coffee. By the definition of the existential quantifier, there exist fierce creatures that do not drink coffee, that is, some fierce creatures do not drink coffee. **23.** The error occurs in step (5), because we cannot assume, as is being done here, that the $c$ that makes $P$ true is the same as the $c$ that makes $Q$ true. **25.** We are given the premises $\forall x(P(x) \rightarrow Q(x))$ and $\neg Q(a)$. We want to show $\neg P(a)$. Suppose, to the contrary, that $\neg P(a)$ is not true. Then $P(a)$ is true. Therefore by universal modus ponens, we have $Q(a)$. But this contradicts the given premise $\neg Q(a)$. Therefore our supposition must have been wrong, and so $\neg P(a)$ is true, as desired.

**27.**

| **Step** | **Reason** |
|---|---|
| 1. $\forall x(P(x) \wedge R(x))$ | Premise |
| 2. $P(a) \wedge R(a)$ | Universal instantiation from (1) |
| 3. $P(a)$ | Simplification from (2) |
| 4. $\forall x(P(x) \rightarrow (Q(x) \wedge S(x)))$ | Premise |
| 5. $Q(a) \wedge S(a)$ | Universal modus ponens from (3) and (4) |
| 6. $S(a)$ | Simplification from (5) |
| 7. $R(a)$ | Simplification from (2) |
| 8. $R(a) \wedge S(a)$ | Conjunction from (7) and (6) |
| 9. $\forall x(R(x) \wedge S(x))$ | Universal generalization from (5) |

**29.**

| **Step** | **Reason** |
|---|---|
| 1. $\exists x \neg P(x)$ | Premise |
| 2. $\neg P(c)$ | Existential instantiation from (1) |
| 3. $\forall x(P(x) \vee Q(x))$ | Premise |
| 4. $P(c) \vee Q(c)$ | Universal instantiation from (3) |
| 5. $Q(c)$ | Disjunctive syllogism from (4) and (2) |
| 6. $\forall x(\neg Q(x) \vee S(x))$ | Premise |

| | |
|---|---|
| 7. $\neg Q(c) \vee S(c)$ | Universal instantiation from (6) |
| 8. $S(c)$ | Disjunctive syllogism from (5) and (7) |
| 9. $\forall x(R(x) \rightarrow \neg S(x))$ | Premise |
| 10. $R(c) \rightarrow \neg S(c)$ | Universal instantiation from (9) |
| 11. $\neg R(c)$ | Modus tollens from (8) and (10) |
| 12. $\exists x \neg R(x)$ | Existential generalization from (11) |

**31.** Let $p$ be "It is raining"; let $q$ be "Yvette has her umbrella"; let $r$ be "Yvette gets wet." Assumptions are $\neg p \vee q, \neg q \vee \neg r$, and $p \vee \neg r$. Resolution on the first two gives $\neg p \vee \neg r$. Resolution on this and the third assumption gives $\neg r$, as desired. **33.** Assume that this proposition is satisfiable. Using resolution on the first two clauses enables us to conclude $q \vee q$; in other words, we know that $q$ has to be true. Using resolution on the last two clauses enables us to conclude $\neg q \vee \neg q$; in other words, we know that $\neg q$ has to be true. This is a contradiction. So this proposition is not satisfiable. **35.** Valid

## Section 1.6

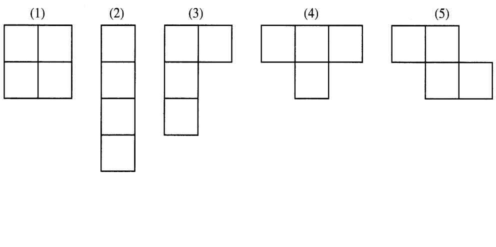
**1.** Let $n = 2k + 1$ and $m = 2l + 1$ be odd integers. Then $n + m = 2(k + l + 1)$ is even. **3.** Suppose that $n$ is even. Then $n = 2k$ for some integer $k$. Therefore, $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Because we have written $n^2$ as 2 times an integer, we conclude that $n^2$ is even. **5.** Direct proof: Suppose that $m + n$ and $n + p$ are even. Then $m + n = 2s$ for some integer $s$ and $n + p = 2t$ for some integer $t$. If we add these, we get $m + p + 2n = 2s + 2t$. Subtracting $2n$ from both sides and factoring, we have $m + p = 2s + 2t - 2n = 2(s + t - n)$. Because we have written $m + p$ as 2 times an integer, we conclude that $m + p$ is even. **7.** Because $n$ is odd, we can write $n = 2k + 1$ for some integer $k$. Then $(k + 1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$. **9.** Suppose that $r$ is rational and $i$ is irrational and $s = r + i$ is rational. Then by Example 7, $s + (-r) = i$ is rational, which is a contradiction. **11.** Because $\sqrt{2} \cdot \sqrt{2} = 2$ is rational and $\sqrt{2}$ is irrational, the product of two irrational numbers is not necessarily irrational. **13.** Proof by contraposition: If $1/x$ were rational, then by definition $1/x = p/q$ for some integers $p$ and $q$ with $q \neq 0$. Because $1/x$ cannot be 0 (if it were, then we'd have the contradiction $1 = x \cdot 0$ by multiplying both sides by $x$), we know that $p \neq 0$. Now $x = 1/(1/x) = 1/(p/q) = q/p$ by the usual rules of algebra and arithmetic. Hence, $x$ can be written as the quotient of two integers with the denominator nonzero. Thus by definition, $x$ is rational. **15.** Assume that it is not true that $x \geq 1$ or $y \geq 1$. Then $x < 1$ and $y < 1$. Adding these two inequalities, we obtain $x + y < 2$, which is the negation of $x + y \geq 2$. **17. a)** Assume that $n$ is odd, so $n = 2k + 1$ for some integer $k$. Then $n^3 + 5 = 2(4k^3 + 6k^2 + 3k + 3)$. Because $n^3 + 5$ is two times some integer, it is even. **b)** Suppose that $n^3 + 5$ is odd and $n$ is odd. Because $n$ is odd and the product of two odd numbers is odd, it follows that $n^2$ is odd and then that $n^3$ is odd. But then $5 = (n^3 + 5) - n^3$ would have to be even because it is the difference of two odd numbers. Therefore, the supposition that $n^3 + 5$ and $n$ were both odd is wrong. **19.** The proposition is vacuously true because 0 is not a positive integer. Vacuous proof. **21.** $P(1)$ is true because $(a + b)^1 = a + b \geq a^1 + b^1 = a + b$. Direct proof. **23.** If we chose 9 or fewer days on each day of the week, this would account for at most $9 \cdot 7 = 63$ days. But we chose 64 days. This contradiction shows that at least 10 of the days we chose must be on the same day of the week. **25.** Suppose by way of contradiction that $a/b$ is a rational root, where $a$ and $b$ are integers and this fraction is in lowest terms (that is, $a$ and $b$ have no common divisor greater than 1). Plug this proposed root into the equation to obtain $a^3/b^3 + a/b + 1 = 0$. Multiply through by $b^3$ to obtain $a^3 + ab^2 + b^3 = 0$. If $a$ and $b$ are both odd, then the left-hand side is the sum of three odd numbers and therefore must be odd. If $a$ is odd and $b$ is even, then the left-hand side is odd + even + even, which is again odd. Similarly, if $a$ is even and $b$ is odd, then the left-hand side is even + even + odd, which is again odd. Because the fraction $a/b$ is in simplest terms, it cannot happen that both $a$ and $b$ are even. Thus in all cases, the left-hand side is odd, and therefore cannot equal 0. This contradiction shows that no such root exists. **27.** First, assume that $n$ is odd, so that $n = 2k + 1$ for some integer $k$. Then $5n + 6 = 5(2k + 1) + 6 = 10k + 11 = 2(5k + 5) + 1$. Hence, $5n + 6$ is odd. To prove the converse, suppose that $n$ is even, so that $n = 2k$ for some integer $k$. Then $5n + 6 = 10k + 6 = 2(5k + 3)$, so $5n + 6$ is even. Hence, $n$ is odd if and only if $5n + 6$ is odd. **29.** This proposition is true. Suppose that $m$ is neither 1 nor $-1$. Then $mn$ has a factor $m$ larger than 1. On the other hand, $mn = 1$, and 1 has no such factor. Hence, $m = 1$ or $m = -1$. In the first case $n = 1$, and in the second case $n = -1$, because $n = 1/m$. **31.** We prove that all these are equivalent to $x$ being even. If $x$ is even, then $x = 2k$ for some integer $k$. Therefore $3x + 2 = 3 \cdot 2k + 2 = 6k + 2 = 2(3k + 1)$, which is even, because it has been written in the form $2t$, where $t = 3k + 1$. Similarly, $x + 5 = 2k + 5 = 2k + 4 + 1 = 2(k + 2) + 1$, so $x + 5$ is odd; and $x^2 = (2k)^2 = 2(2k^2)$, so $x^2$ is even. For the converses, we will use a proof by contraposition. So assume that $x$ is not even; thus $x$ is odd and we can write $x = 2k + 1$ for some integer $k$. Then $3x + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$, which is odd (i.e., not even), because it has been written in the form $2t + 1$, where $t = 3k + 2$. Similarly, $x + 5 = 2k + 1 + 5 = 2(k + 3)$, so $x + 5$ is even (i.e., not odd). That $x^2$ is odd was already proved in Example 1. **33.** We give proofs by contraposition of $(i) \rightarrow (ii)$, $(ii) \rightarrow (i)$, $(i) \rightarrow (iii)$, and $(iii) \rightarrow (i)$. For the first of these, suppose that $3x + 2$ is rational, namely, equal to $p/q$ for some integers $p$ and $q$ with $q \neq 0$. Then we can write $x = ((p/q) - 2)/3 = (p - 2q)/(3q)$, where $3q \neq 0$. This shows that $x$ is rational. For the second conditional statement, suppose that $x$ is rational, namely, equal to $p/q$ for some integers $p$ and $q$ with $q \neq 0$. Then we can write $3x + 2 = (3p + 2q)/q$, where $q \neq 0$. This shows that $3x + 2$ is rational. For the third conditional statement, suppose that $x/2$ is rational, namely, equal to $p/q$ for some integers $p$ and $q$ with $q \neq 0$. Then we can write $x = 2p/q$, where $q \neq 0$. This shows that $x$ is rational. And for the fourth conditional statement, suppose that $x$ is rational, namely, equal to $p/q$ for some integers $p$ and $q$ with $q \neq 0$. Then we can write $x/2 = p/(2q)$, where $2q \neq 0$. This shows that $x/2$ is rational.

**35.** No **37.** Suppose that $p_1 \to p_4 \to p_2 \to p_5 \to p_3 \to p_1$. To prove that one of these propositions implies any of the others, just use hypothetical syllogism repeatedly. **39.** We will give a proof by contradiction. Suppose that $a_1, a_2, \ldots, a_n$ are all less than $A$, where $A$ is the average of these numbers. Then $a_1 + a_2 + \cdots + a_n < nA$. Dividing both sides by $n$ shows that $A = (a_1 + a_2 + \cdots + a_n)/n < A$, which is a contradiction. **41.** We will show that the four statements are equivalent by showing that $(i)$ implies $(ii)$, $(ii)$ implies $(iii)$, $(iii)$ implies $(iv)$, and $(iv)$ implies $(i)$. First, assume that $n$ is even. Then $n = 2k$ for some integer $k$. Then $n + 1 = 2k + 1$, so $n + 1$ is odd. This shows that $(i)$ implies $(ii)$. Next, suppose that $n + 1$ is odd, so $n + 1 = 2k + 1$ for some integer $k$. Then $3n + 1 = 2n + (n + 1) = 2(n + k) + 1$, which shows that $3n + 1$ is odd, showing that $(ii)$ implies $(iii)$. Next, suppose that $3n + 1$ is odd, so $3n + 1 = 2k + 1$ for some integer $k$. Then $3n = (2k + 1) - 1 = 2k$, so $3n$ is even. This shows that $(iii)$ implies $(iv)$. Finally, suppose that $n$ is not even. Then $n$ is odd, so $n = 2k + 1$ for some integer $k$. Then $3n = 3(2k + 1) = 6k + 3 = 2(3k + 1) + 1$, so $3n$ is odd. This completes a proof by contraposition that $(iv)$ implies $(i)$.

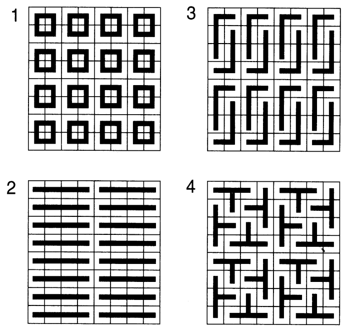
## Section 1.7

**1.** $1^2 + 1 = 2 \geq 2 = 2^1$; $2^2 + 1 = 5 \geq 4 = 2^2$; $3^2 + 1 = 10 \geq 8 = 2^3$; $4^2 + 1 = 17 \geq 16 = 2^4$ **3.** If $x \leq y$, then $\max(x, y) + \min(x, y) = y + x = x + y$. If $x \geq y$, then $\max(x, y) + \min(x, y) = x + y$. Because these are the only two cases, the equality always holds. **5.** There are four cases. *Case 1:* $x \geq 0$ and $y \geq 0$. Then $|x| + |y| = x + y = |x + y|$. *Case 2:* $x < 0$ and $y < 0$. Then $|x| + |y| = -x + (-y) = -(x + y) = |x + y|$ because $x + y < 0$. *Case 3:* $x \geq 0$ and $y < 0$. Then $|x| + |y| = x + (-y)$. If $x \geq -y$, then $|x + y| = x + y$. But because $y < 0$, $-y > y$, so $|x| + |y| = x + (-y) > x + y = |x + y|$. If $x < -y$, then $|x + y| = -(x + y) = -x + (-y)$. But because $x \geq 0$, $x \geq -x$, so $|x| + |y| = x + (-y) \geq -x + (-y) = |x + y|$. *Case 4:* $x < 0$ and $y \geq 0$. Identical to Case 3 with the roles of $x$ and $y$ reversed. **7.** 10,001, 10,002, …, 10,100 are all nonsquares, because $100^2 = 10{,}000$ and $101^2 = 10{,}201$; constructive. **9.** $8 = 2^3$ and $9 = 3^2$ **11.** Let $x = 2$ and $y = \sqrt{2}$. If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, then let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$. Then $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = 2^{1/2} = \sqrt{2}$. **13. a)** This statement asserts the existence of $x$ with a certain property. If we let $y = x$, then we see that $P(x)$ is true. If $y$ is anything other than $x$, then $P(x)$ is not true. Thus, $x$ is the unique element that makes $P$ true. **b)** The first clause here says that there is an element that makes $P$ true. The second clause says that whenever two elements both make $P$ true, they are in fact the same element. Together these say that $P$ is satisfied by exactly one element. **c)** This statement asserts the existence of an $x$ that makes $P$ true and has the further property that whenever we find an element that makes $P$ true, that element is $x$. In other words, $x$ is the unique element that makes $P$ true. **15.** The equation $|a - c| = |b - c|$ is equivalent to the disjunction of two equations: $a - c = b - c$ or $a - c = -b + c$. The first of these is equivalent to $a = b$, which contradicts the assumptions made in this problem, so the original equation is equivalent to $a - c = -b + c$. By adding $b + c$ to both sides and dividing by 2, we see that this equation is equivalent to $c = (a + b)/2$. Thus, there is a unique solution. Furthermore, this $c$ is an integer, because the sum of the odd integers $a$ and $b$ is even. **17.** We are being asked to solve $n = (k - 2) + (k + 3)$ for $k$. Using the usual, reversible, rules of algebra, we see that this equation is equivalent to $k = (n - 1)/2$. In other words, this is the one and only value of $k$ that makes our equation true. Because $n$ is odd, $n - 1$ is even, so $k$ is an integer. **19.** If $x$ is itself an integer, then we can take $n = x$ and $\epsilon = 0$. No other solution is possible in this case, because if the integer $n$ is greater than $x$, then $n$ is at least $x + 1$, which would make $\epsilon \geq 1$. If $x$ is not an integer, then round it up to the next integer, and call that integer $n$. Let $\epsilon = n - x$. Clearly $0 \leq \epsilon < 1$; this is the only $\epsilon$ that will work with this $n$, and $n$ cannot be any larger, because $\epsilon$ is constrained to be less than 1. **21.** The harmonic mean of distinct positive real numbers $x$ and $y$ is always less than their geometric mean. To prove $2xy/(x + y) < \sqrt{xy}$, multiply both sides by $(x + y)/(2\sqrt{xy})$ to obtain the equivalent inequality $\sqrt{xy} < (x + y)/2$, which is proved in Example 14. **23.** The parity (oddness or evenness) of the sum of the numbers written on the board never changes, because $j + k$ and $|j - k|$ have the same parity (and at each step we reduce the sum by $j + k$ but increase it by $|j - k|$). Therefore the integer at the end of the process must have the same parity as $1 + 2 + \cdots + (2n) = n(2n + 1)$, which is odd because $n$ is odd. **25.** Without loss of generality we can assume that $n$ is nonnegative, because the fourth power of an integer and the fourth power of its negative are the same. We divide an arbitrary positive integer $n$ by 10, obtaining a quotient $k$ and remainder $l$, whence $n = 10k + l$, and $l$ is an integer between 0 and 9, inclusive. Then we compute $n^4$ in each of these 10 cases. We get the following values, where $X$ is some integer that is a multiple of 10, whose exact value we do not care about. $(10k + 0)^4 = 10{,}000k^4 = 10{,}000k^4 + 0$, $(10k + 1)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 1$, $(10k + 2)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 16$, $(10k + 3)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 81$, $(10k + 4)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 256$, $(10k + 5)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 625$, $(10k + 6)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 1296$, $(10k + 7)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 2401$, $(10k + 8)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 4096$, $(10k + 9)^4 = 10{,}000k^4 + X \cdot k^3 + X \cdot k^2 + X \cdot k + 6561$. Because each coefficient indicated by $X$ is a multiple of 10, the corresponding term has no effect on the ones digit of the answer. Therefore the ones digits are 0, 1, 6, 1, 6, 5, 6, 1, 6, 1, respectively, so it is always a 0, 1, 5, or 6. **27.** Because $n^3 > 100$ for all $n > 4$, we need only note that $n = 1, n = 2, n = 3$, and $n = 4$ do not satisfy $n^2 + n^3 = 100$. **29.** Because $5^4 = 625$, both $x$ and $y$ must be less than 5. Then $x^4 + y^4 \leq 4^4 + 4^4 = 512 < 625$. **31.** If it is not true that $a \leq \sqrt[3]{n}$, $b \leq \sqrt[3]{n}$, or $c \leq \sqrt[3]{n}$, then $a > \sqrt[3]{n}$, $b > \sqrt[3]{n}$, and $c > \sqrt[3]{n}$. Multiplying these inequalities of positive numbers together we obtain $abc < (\sqrt[3]{n})^3 = n$, which implies the negation of our hypothesis that $n = abc$. **33.** By finding a common denominator, we can assume that the given rational numbers are $a/b$ and $c/b$, where $b$ is a positive integer and $a$ and $c$ are integers with $a < c$. In particular, $(a+1)/b \le c/b$. Thus, $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, $x$ is irrational, because if $x$ were rational, then $2(bx - a) = \sqrt{2}$ would be as well, in violation of Example 10 in Section 1.6. **35. a)** Without loss of generality, we can assume that the $x$ sequence is already sorted into nondecreasing order, because we can relabel the indices. There are only a finite number of possible orderings for the $y$ sequence, so if we can show that we can increase the sum (or at least keep it the same) whenever we find $y_i$ and $y_j$ that are out of order (i.e., $i < j$ but $y_i > y_j$) by switching them, then we will have shown that the sum is largest when the $y$ sequence is in nondecreasing order. Indeed, if we perform the swap, then we have added $x_i y_j + x_j y_i$ to the sum and subtracted $x_i y_i + x_j y_j$. The net effect is to have added $x_i y_j + x_j y_i - x_i y_i - x_j y_j = (x_j - x_i)(y_i - y_j)$, which is nonnegative by our ordering assumptions. **b)** Similar to part (a) **37. a)** $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ **b)** $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ **c)** $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$ **d)** $21 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$ **39.** Without loss of generality, assume that the upper left and upper right corners of the board are removed. Place three dominoes horizontally to fill the remaining portion of the first row, and fill each of the other seven rows with four horizontal dominoes. **41.** Because there is an even number of squares in all, either there is an even number of squares in each row or there is an even number of squares in each column. In the former case, tile the board in the obvious way by placing the dominoes horizontally, and in the latter case, tile the board in the obvious way by placing the dominoes vertically. **43.** We can rotate the board if necessary to make the removed squares be 1 and 16. Square 2 must be covered by a domino. If that domino is placed to cover squares 2 and 6, then the following domino placements are forced in succession: 5-9, 13-14, and 10-11, at which point there is no way to cover square 15. Otherwise, square 2 must be covered by a domino placed at 2-3. Then the following domino placements are forced: 4-8, 11-12, 6-7, 5-9, and 10-14, and again there is no way to cover square 15. **45.** Remove the two black squares adjacent to a white corner, and remove two white squares other than that corner. Then no domino can cover that white corner.

**47. a)**




**b)** The picture shows tilings for the first four patterns.




To show that pattern 5 cannot tile the checkerboard, label the squares from 1 to 64, one row at a time from the top, from left to right in each row. Thus, square 1 is the upper left corner, and square 64 is the lower right. Suppose we did have a tiling. By symmetry and without loss of generality, we may suppose that the tile is positioned in the upper left corner, covering squares 1, 2, 10, and 11. This forces a tile to be adjacent to it on the right, covering squares 3, 4, 12, and 13. Continue in this manner and we are forced to have a tile covering squares 6, 7, 15, and 16. This makes it impossible to cover square 8. Thus, no tiling is possible.

## Supplementary Exercises

**1. a)** $q \to p$ **b)** $q \wedge p$ **c)** $\neg q \vee \neg p$ **d)** $q \leftrightarrow p$ **3. a)** The proposition cannot be false unless $\neg p$ is false, so $p$ is true. If $p$ is true and $q$ is true, then $\neg q \wedge (p \to q)$ is false, so the conditional statement is true. If $p$ is true and $q$ is false, then $p \to q$ is false, so $\neg q \wedge (p \to q)$ is false and the conditional statement is true. **b)** The proposition cannot be false unless $q$ is false. If $q$ is false and $p$ is true, then $(p \vee q) \wedge \neg p$ is false, and the conditional statement is true. If $q$ is false and $p$ is false, then $(p \vee q) \wedge \neg p$ is false, and the conditional statement is true. **5.** $\neg q \to \neg p$; $p \to q$; $\neg p \to \neg q$ **7.** $(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$ **9.** Translating these statements into symbols, using the obvious letters, we have $\neg t \to \neg g$, $\neg g \to \neg q$, $r \to q$, and $\neg t \wedge r$. Assume the statements are consistent. The fourth statement tells us that $\neg t$ must be true. Therefore by modus ponens with the first statement, we know that $\neg g$ is true, hence (from the second statement), that $\neg q$ is true. Also, the fourth statement tells us that $r$ must be true, and so again modus ponens (third statement) makes $q$ true. This is a contradiction: $q \wedge \neg q$. Thus the statements are inconsistent. **11.** Brenda **13.** The premises cannot both be true, because they are contradictory. Therefore it is (vacuously) true that whenever all the premises are true, the conclusion is also true, which by definition makes this a valid argument. Because the premises are not both true, we cannot conclude that the conclusion is true. **15. a)** F **b)** T **c)** F **d)** T **e)** F **f)** T **17.** Many answers are possible. One example is United States senators.

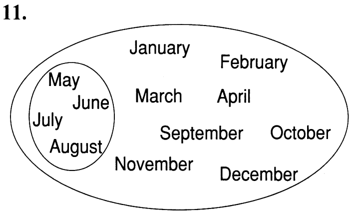
**19.** $\forall x\exists y\exists z(y \neq z \land \forall w(P(w,x) \leftrightarrow (w = y \lor w = z)))$ **21. a)** $\neg\exists x P(x)$ **b)** $\exists x(P(x) \land \forall y(P(y) \to y = x))$ **c)** $\exists x_1\exists x_2(P(x_1) \land P(x_2) \land x_1 \neq x_2 \land \forall y(P(y) \to (y = x_1 \lor y = x_2)))$ **d)** $\exists x_1 \exists x_2 \exists x_3(P(x_1) \land P(x_2) \land P(x_3) \land x_1 \neq x_2 \land x_1 \neq x_3 \land x_2 \neq x_3 \land \forall y(P(y) \to (y = x_1 \lor y = x_2 \lor y = x_3)))$ **23.** Suppose that $\exists x(P(x) \to Q(x))$ is true. Then either $Q(x_0)$ is true for some $x_0$, in which case $\forall x P(x) \to \exists x\, Q(x)$ is true; or $P(x_0)$ is false for some $x_0$, in which case $\forall x P(x) \to \exists x Q(x)$ is true. Conversely, suppose that $\exists x(P(x) \to Q(x))$ is false. That means that $\forall x(P(x) \land \neg Q(x))$ is true, which implies $\forall x P(x)$ and $\forall x(\neg Q(x))$. This latter proposition is equivalent to $\neg\exists x Q(x)$. Thus, $\forall x P(x) \to \exists x Q(x)$ is false. **25.** No **27.** $\forall x\, \forall z\, \exists y\, T(x, y, z)$, where $T(x, y, z)$ is the statement that student $x$ has taken class $y$ in department $z$, where the domains are the set of students in the class, the set of courses at this university, and the set of departments in the school of mathematical sciences **29.** $\exists! x \exists! y\, T(x, y)$ and $\exists x \forall z((\exists y \forall w(T(z, w) \leftrightarrow w = y)) \leftrightarrow z = x)$, where $T(x, y)$ means that student $x$ has taken class $y$ and the domain is all students in this class **31.** $P(a) \to Q(a)$ and $Q(a) \to R(a)$ by universal instantiation; then $\neg Q(a)$ by modus tollens and $\neg P(a)$ by modus tollens **33.** We give a proof by contraposition and show that if $\sqrt{x}$ is rational, then $x$ is rational, assuming throughout that $x \geq 0$. Suppose that $\sqrt{x} = p/q$ is rational, $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational ($q^2$ is again nonzero). **35.** We can give a constructive proof by letting $m = 10^{500} + 1$. Then $m^2 = (10^{500} + 1)^2 > (10^{500})^2 = 10^{1000}$. **37.** 23 cannot be written as the sum of eight cubes. **39.** 223 cannot be written as the sum of 36 fifth powers.

# CHAPTER 2

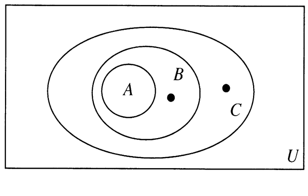
## Section 2.1

**1. a)** $\{-1, 1\}$ **b)** $\{1,2,3,4,5,6,7,8,9,10,11\}$ **c)** $\{0,1,4,9,16,25,36,49,64,81\}$ **d)** $\emptyset$ **3. a)** Yes **b)** No **c)** No **5. a)** Yes **b)** No **c)** Yes **d)** No **e)** No **f)** No **7. a)** False **b)** False **c)** False **d)** True **e)** False **f)** False **g)** True **9. a)** True **b)** True **c)** False **d)** True **e)** True **f)** False

**11.**




**13.** The dots in certain regions indicate that those regions are not empty.




**15.** Suppose that $x \in A$. Because $A \subseteq B$, this implies that $x \in B$. Because $B \subseteq C$, we see that $x \in C$. Because $x \in A$ implies that $x \in C$, it follows that $A \subseteq C$. **17. a)** 1 **b)** 1 **c)** 2 **d)** 3 **19. a)** $\{\emptyset, \{a\}\}$ **b)** $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ **c)** $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$ **21. a)** 8 **b)** 16 **c)** 2 **23. a)** $\{(a, y), (b, y), (c, y), (d, y), (a, z), (b, z), (c, z), (d, z)\}$ **b)** $\{(y, a), (y, b), (y, c), (y, d), (z, a), (z, b), (z, c), (z, d)\}$ **25.** The set of triples $(a, b, c)$, where $a$ is an airline and $b$ and $c$ are cities **27.** $\emptyset \times A = \{(x, y) \mid x \in \emptyset$ and $y \in A\} = \emptyset = \{(x, y) \mid x \in A$ and $y \in \emptyset\} = A \times \emptyset$ **29.** $mn$ **31.** The elements of $A \times B \times C$ consist of 3-tuples $(a, b, c)$, where $a \in A$, $b \in B$, and $c \in C$, whereas the elements of $(A \times B) \times C$ look like $((a, b), c)$—ordered pairs, the first coordinate of which is again an ordered pair. **33. a)** The square of a real number is never $-1$. True **b)** There exists an integer whose square is 2. False **c)** The square of every integer is positive. False **d)** There is a real number equal to its own square. True **35. a)** $\{-1, 0, 1\}$ **b)** $\mathbf{Z} - \{0, 1\}$ **c)** $\emptyset$ **37.** We must show that $\{\{a\}, \{a, b\}\} = \{\{c\}, \{c, d\}\}$ if and only if $a = c$ and $b = d$. The "if" part is immediate. So assume these two sets are equal. First, consider the case when $a \neq b$. Then $\{\{a\}, \{a, b\}\}$ contains exactly two elements, one of which contains one element. Thus, $\{\{c\}, \{c, d\}\}$ must have the same property, so $c \neq d$ and $\{c\}$ is the element containing exactly one element. Hence, $\{a\} = \{c\}$, which implies that $a = c$. Also, the two-element sets $\{a, b\}$ and $\{c, d\}$ must be equal. Because $a = c$ and $a \neq b$, it follows that $b = d$. Second, suppose that $a = b$. Then $\{\{a\}, \{a, b\}\} = \{\{a\}\}$, a set with one element. Hence, $\{\{c\}, \{c, d\}\}$ has only one element, which can happen only when $c = d$, and the set is $\{\{c\}\}$. It then follows that $a = c$ and $b = d$. **39.** Let $S = \{a_1, a_2, \ldots, a_n\}$. Represent each subset of $S$ with a bit string of length $n$, where the $i$th bit is 1 if and only if $a_i \in S$. To generate all subsets of $S$, list all $2^n$ bit strings of length $n$ (for instance, in increasing order), and write down the corresponding subsets.

## Section 2.2

**1. a)** The set of students who live within one mile of school and who walk to classes **b)** The set of students who live within one mile of school or who walk to classes (or who do both) **c)** The set of students who live within one mile of school but do not walk to classes **d)** The set of students who walk to classes but live more than one mile away from school **3. a)** $\{0,1,2,3,4,5,6\}$ **b)** $\{3\}$ **c)** $\{1, 2, 4, 5\}$ **d)** $\{0, 6\}$ **5.** $\overline{\overline{A}} = \{x \mid \neg(x \in \overline{A})\} = \{x \mid \neg(\neg x \in A)\} = \{x \mid x \in A\} = A$ **7. a)** $A \cup U = \{x \mid x \in A \lor x \in U\} = \{x \mid x \in A \lor \mathbf{T}\} = \{x \mid \mathbf{T}\} = U$ **b)** $A \cap \emptyset = \{x \mid x \in A \land x \in \emptyset\} = \{x \mid x \in$